New Multigrid Solver Advances

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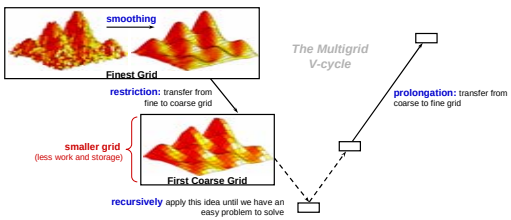
Abstract: We present two recent multigrid advances. First, we provide promising numerical results for the application of adaptive algebraic multigrid to linear systems with oscillatory low-energy modes. Second we introduce the idea of compatible relaxation as a main tool in selecting the coarse degrees of freedom.

Introduction: multigrid methods use coarse grids to efficiently damp out smooth error



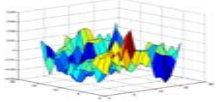
Algebraic Multigrid (AMG) methods are based on the underlying matrix, and little else

- **AMG is applicable to unstructured-grid problems**
- **Characterizing *algebraically smooth error* (error not damped by relaxation) is crucial for designing robust AMG methods**
- **Algebraically smooth error need not be geometrically smooth!**

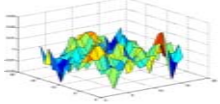
Application I: AMG for Quantum Chromodynamics (QCD)

- **Lowest eigenmode of *H*H***

Real part



Imaginary part



$\beta = 6$ on 16 x 16 lattice

- **Challenges:**
  - Problem becomes nearly singular
  - H has multiple (locally distinct) near nullspace components
  - These components are unknown and oscillatory!

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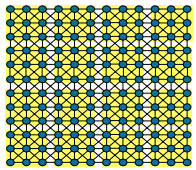
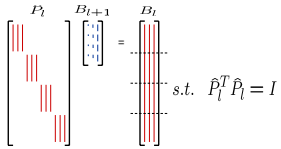
Method: Adaptive Smoothed Aggregation (αSA)

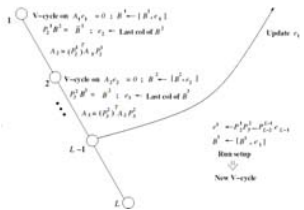
- **Points are aggregated**
- **A tentative prolongator is constructed from aggregates and local basis functions**
- **Tentative prolongation is defined by "chopping up" the representatives**
  - Local QR factorization is used to orthonormalize
  - Can have arbitrary numbers of DOFs per aggregate





$$P_l = S_l \hat{P}_l$$

- **Main idea:** uncover the slowly-converging error components by applying the "current method" to the system $Ax = 0$ and use them to adapt (improve) the method



Numerical Results: αSA and QCD

- **2D Hermitian Dirac-Wilson Formulation**
  - Solve the 2 x 2 equivalent real system
  - Software *SAMIS.dat(amg)* (M. Brezina), GS smoother and V(2,2) cycles

αAMG-PCG, $\beta = 3$, $r = 8$; $m_g = \rho - \rho_{crit}$

n / m_g	0.001	0.01	0.05	0.1	0.3
4096	0.32	0.29	0.27	0.26	0.27
16384	0.50	0.42	0.42	0.40	0.31
65536	0.48	0.37	0.36	0.36	0.30

αAMG-PCG / CG, $n = 16384$, $L = 3$, $r = 8$

β / m_g	0.001	0.01	0.05	0.1	0.3
2	.37 / .99	.33 /.99	.31 /.96	.31 /.94	.31 /.85
3	.50 / .99	.42 /.98	.42 /.97	.40 /.93	.31 /.86
5	.38 / .99	.31 /.99	.31 /.96	.29 /.92	.28 /.83

Application II: AMG coarsening via compatible relaxation

Theory

- **Consider solving** $Au = f$, A **SPD**

$$e_{k+1} = (I - M^{-1}A)e_k$$

- **Consider smoothers with error propagation**

$$(I - \widetilde{M}^{-1}A) = (I - M^{-T}A)(I - M^{-1}A)$$

- **Denote the symmetrized smoother operator by**

$$\|E_{TG}\|_A^2 \le 1 - \frac{1}{K}; \qquad K = \sup_e \frac{\|(I - PR)e\|_{\widetilde{M}}^2}{\|e\|_A^2}$$

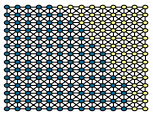
- **Theorem:** two-grid convergence

$$K \le \eta K_\star; \quad \eta = \|PR\|_A; \quad K_\star = \inf_P \sup_e \frac{\|(I - PR)e\|_{\widetilde{M}}^2}{\|e\|_A^2}$$

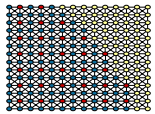
- **Define R so that it does not depend on interpolation P**
  - Defines the coarse-grid variables, $u_c = Ru$
  - Example: R=[0, I] (P^T=[W^T, I]^T), i.e., subset of the fine grid
- **Theorem:** splits coarse-grid correction into two parts
  (1) Small $K_\star$ insures coarse grid quality – use CR
  (2) Small η insures interpolation quality – necessary; does not depend on relaxation!

- **We defined several general *CR* methods and showed that fast convergence implies a good coarse grid**

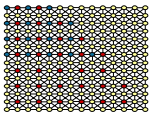
Method: Selecting Coarse Grids via CR



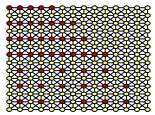
Step 1: Run *CR*; check convergence



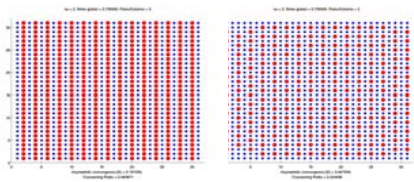
Step 2: Ind. Set



Step 3: Run *CR*; check convergence



Step 4: Final *C*-pts

Numerical Results: Anisotropic 9-pt FE… coarse grids reflect smoother used in *CR*

- Pointwise Gauss-Seidel *CR* ($\rho_{cr} = 0.19$) vs. Line Jacobi *CR* ($\rho_{cr} = 0.45$)
- Not possible with standard definitions of strength of connection!